EE 440 Photonic Systems & Technologies 2025 – Série 1
SOLUTIONS Graded Exercise

Einstein coefficients – In class we have derived the probability densities of spontaneous and stimulated transitions under the assumption of monochromatic light having a photon flux density ϕ, such that the *probability density of an induced transition* was $W_i = \phi\sigma(\nu)$.

We will here carry similar analysis but with broadband light as done by Einstein in 1917, where he assumed energy exchange between atoms and radiation at thermal equilibrium through broadband radiation of spectral energy density $\varrho(\nu)$ (unit of $\mathrm{Jm^{-3}s}$) such as coming from a black body. Considering two levels 1 and 2, with 2 being an upper level, he named:

- the probability density of transition due to *spontaneous emission* A_{21}. Unit of A_{21} is s^{-1}.
- the *probability density of stimulated emissions* B_{12} and B_{21}, depending on the direction of the transition. Unit of B_{12} and B_{21} is $\mathrm{J^{-1}m^3s^{-2}}$

a. Considering the three possible atom-radiation interactions, write the rate equation for the population of the upper level N_2 $(i.e. \frac{dN_2}{dt})$ using Einstein *A* and *B* coefficients and assuming spectral energy density $\varrho(\nu)$. (Start by expressing the rate of change of N_2 for the three processes)

Spontaneous emission (decrease in population): $\left(\frac{dN_2}{dt}\right)_{spon} = -A_{21}N_2$

Stimulated emission (decrease in population): $\left(\frac{dN_2}{dt}\right)_{stim} = -B_{21}\,\varrho(\nu)N_2$

Absorption (increase in population): $\left(\frac{dN_2}{dt}\right)_{abs} = B_{12}\,\varrho(\nu)N_1$

The rate equation is therefore:

$$\boldsymbol{\frac{dN_2}{dt} = -A_{21}N_2 - B_{21}\,\varrho(\nu)N_2 + B_{12}\,\varrho(\nu)N_1}$$

b. Given that the blackbody spectrum of radiation is given by $\varrho(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp\left(\frac{h\nu}{k_BT}\right)-1}$, and using Boltzmann distribution, show that at steady state $B_{12} = B_{21} \equiv B$ and that $\frac{A_{21}}{B} = \frac{8\pi h\nu^3}{c^3}$.

At steady state:

$$\frac{dN_2}{dt} = -A_{21}N_2 - B_{21}\,\varrho(\nu)N_2 + B_{12}\,\varrho(\nu)N_1 = 0$$

$$\varrho(\nu)(B_{12}N_1 - B_{21}N_2) = A_{21}N_2$$

$$\varrho(\nu) = \frac{A_{21}N_2}{(B_{12}N_1 - B_{21}N_2)}$$

$$\varrho(\nu) = \frac{A_{21}}{\left(B_{12}\frac{N_1}{N_2} - B_{21}\right)}$$

$$\varrho(\nu) = \frac{\frac{A_{21}}{B_{21}}}{\left(\frac{B_{12}}{B_{21}}\frac{N_1}{N_2} - 1\right)}$$

We know from Boltzmann distribution that $\frac{N_1}{N_2} = \exp\left(\frac{E_2 - E_1}{k_B T}\right) = \exp\left(\frac{h\nu}{k_{BT}}\right)$

Therefore

$$\varrho(\nu) = \frac{A_{21}}{B_{21}}\frac{1}{\left(\frac{B_{12}}{B_{21}}\exp\left(\frac{h\nu}{k_{BT}}\right) - 1\right)}$$

But we are given that :

$$\varrho(\nu) = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp\left(\frac{h\nu}{k_{BT}}\right) - 1}$$

By inspection we conclude that

$$\boldsymbol{\frac{B_{12}}{B_{21}} = 1 \Rightarrow B_{21} = B_{12}}$$

$$\boldsymbol{\frac{A_{21}}{B_{21}} = \frac{8\pi h\nu^3}{c^3}}$$

c. Based on the answer found in b. show that the rate of change of N_2 due to spontaneous emission always dominates the rate of change of N_2 due to stimulated emission at room temperature (T = 300 K and $k_B T$ = 25 meV) in the near infrared region where $h\nu \approx 1\ eV$. This is another way of understanding why lasing cannot occur under such condition and why pumping is required.

$$\left(\frac{dN_2}{dt}\right)_{spon} = -A_{21}N_2$$

$$\left(\frac{dN_2}{dt}\right)_{stim} = -B_{21}\,\varrho(\nu)N_2$$

Therefore:

$$\frac{\left(\frac{dN_2}{dt}\right)_{stim}}{\left(\frac{dN_2}{dt}\right)_{spon}} = \frac{B_{21}}{A_{21}}\varrho(\nu) = \frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1} = \frac{1}{\exp\left(\frac{1}{0.025}\right) - 1} \ll 1$$

Rate of stimulated emission is negligible compared to spontaneous emission.

d. At what wavelength are the rates of spontaneous and stimulated emission equal in a two-level system at room temperature (T = 300 K)?

We need to find when :

$$\left(\frac{dN_2}{dt}\right)_{spon} = \left(\frac{dN_2}{dt}\right)_{stim}$$

Meaning when :

$$-A_{21}N_2 = -B_{21}\,\varrho(\nu)N_2$$

$$\frac{A_{21}}{B_{21}} = \varrho(\nu)$$

$$\frac{8\pi h\nu^3}{c^3} = \frac{8\pi h\nu^3}{c^3}\frac{1}{\exp\left(\frac{h\nu}{k_B T}\right) - 1}$$

$$\exp\left(\frac{h\nu}{k_B T}\right) = 2$$

$$\nu = \ln(2)\frac{k_B T}{h}$$

$$\nu = 4.33 \cdot 10^{12}\ \mathrm{Hz}$$

$$\boldsymbol{\lambda = 69.21\ \mu m}$$